HW 11 SOLUTIONS

SECTION 5.1

(1) $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0.$

$\lambda = 0 \Rightarrow y = Ax + B$

$y' = A = 0$ AT $x = \pi$

$y = B = 0$ AT $x = 0$

$\Rightarrow y = 0$ X NO GOOD; E-FUNCTION CAN'T $= 0$.

$\lambda > 0$

$\Rightarrow \lambda = +\mu^2 > 0.$

$y'' + \mu^2 y = 0 \Rightarrow y = A\sin\mu x + B\cos\mu x$

$y(0) = B = 0$

$y'(x) = \cancel{\mu A \sin(\mu x)}\ \mu A\cos(\mu x), \quad y'(\pi) = \cancel{\mu A \sin(\mu\pi)}\ \mu A\cos(\mu\pi) = 0$

SO EITHER $\mu = 0 \Rightarrow \lambda = 0$, WHICH GIVES $y = 0$ AS ABOVE X

OR $A = 0 \Rightarrow y = 0$ X

OR $\cancel{\sin(\mu\pi)}\ \cos(\mu\pi) = 0 \Rightarrow \mu\pi = \cancel{n\pi}\ n\pi + \pi/2$ FOR ANY INTEGER n.

$\Rightarrow \cancel{\mu = n}\ \mu = n + 1/2$

$\therefore$ $\boxed{\text{EIGENVALUES ARE } \lambda_n = \mu^2 = (n+1/2)^2 \text{ WITH ASSOCIATED EIGENFUNCTIONS } \cancel{y_n(x) = \sin(nx)}\ y_n = \sin((n+1/2)x)} \quad \boxed{n \geq 0}$

3) $y'' + \lambda y = 0 \quad , \quad y'(0) = 0 \quad , \quad y'(\pi) = 0.$

$\underline{\lambda = 0}$ $\Rightarrow y = Ax + B$

$y'(x) = A \qquad y'(0) = A = 0$ ✓

$y'(\pi) = 0 = A$ ✓

$\Rightarrow y = B$

$\therefore$ $\boxed{\lambda = 0 \text{ IS AN E-VAL w/ EIGENFUNCTION } y = 1}$

RECALL THAT E-FUNCTIONS ARE ONLY DEFINED UP TO AN ARBITRARY CONSTANT.

$\underline{\lambda < 0}$: $\lambda = -\mu^2 < 0 \quad \Rightarrow y'' - \mu^2 y = 0$

$\Rightarrow \quad y = Ae^{\mu x} + Be^{-\mu x}$

$y'(x) = \mu A e^{\mu x} - \mu B e^{-\mu x}$

$y'(0) = \mu A - \mu B = 0 \quad \Rightarrow A = B$

$y'(x) = A(\mu e^{\mu x} - \mu e^{-\mu x})$

$y'(\pi) = 0 = A(\underbrace{\mu e^{\mu\pi} - \mu e^{-\mu\pi}}_{\text{JUST A REAL NUMBER}}) \quad \Rightarrow A = 0.$

$\Rightarrow B = 0.$

$\Rightarrow y = 0$ X NOT AN E-FUNCTION

$\underline{\lambda < 0}$: $\lambda = +\mu^2 > 0 \quad \Rightarrow \quad y(x) = A\cos(\mu x) + B\sin(\mu x)$

$y'(x) = -A\mu \sin(\mu x) + B\mu \cos(\mu x) \quad \Rightarrow y'(0) = -A\mu = 0 \quad \Rightarrow A = 0.$

$y'(\pi) = B\mu\cos(\mu\pi) = 0 \quad \Rightarrow \cos(\mu\pi) = 0 \quad \Rightarrow \mu\pi = \pi/2 + n\pi$

$\Rightarrow \mu = \frac{1}{2} + n$ $\quad \therefore$ $\lambda_n = (1/2 + n)^2,\ y_n(x) = \sin((1/2 + n)x)$ IS E-VAL/E-FUNCTION PAIR.

SECTION 5.2

(2) $\sin\left(\frac{\pi x}{L}\right)$ FIND THE FUNDAMENTAL PERIOD.

(WE KNOW SINE IS A PERIODIC FUNCTION).

FUNDAMENTAL PERIOD ~~GIVEN~~ IS SMALLEST NUMBER γ SUCH THAT

$f(x+\gamma) = f(x)$ HOLDS FOR ALL x.

$\sin\theta$ HAS FUNDAMENTAL PERIOD $2\pi = \gamma \Rightarrow$ WHEN $\theta = \frac{\pi x}{L}$,

$\sin\left(\frac{\pi x}{L}\right)$ HAS FUNDAMENTAL PERIOD

~~$\sin\left(\frac{\pi x}{L}\right)$ HAS FUNDAMENTAL PERIOD~~

$$\frac{\pi \gamma}{L} = 2\pi$$

$$\Rightarrow \boxed{\gamma = 2L}$$

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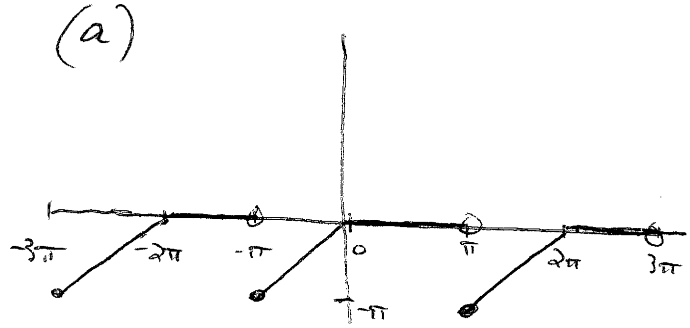
(5) $f(x) = \begin{cases} x & -\pi \le x < 0 \\ 0 & 0 \le x < \pi \end{cases}$

HERE $L = \pi$

(a)



(b)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) + b_n \sin\left(\frac{n\pi x}{\pi}\right)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x), \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \quad b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x \, dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} = \frac{1}{\pi}\cdot\frac{-1}{2}\cdot\pi^2 = -\frac{\pi}{2}.$$

$$\begin{array}{ccc} u & & dv \\ x & & \cos(nx) \\ & + & \\ 1 & & \frac{1}{n}\sin(nx) \\ & - & \\ 0 & & -\frac{1}{n^2}\cancel{\sin}\cos(nx) \end{array}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cdot\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\sin(nx)\cdot x + \frac{1}{n^2}\cos(nx)\right]_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\cos(-n\pi)\right]$$

$$\cos(-n\pi) = \cos(n\pi) = (-1)^n.$$

$$= \frac{1}{\pi}\,\frac{1}{n^2}\left[1 - (-1)^n\right]$$

$$\begin{array}{ccc} u & & dv \\ x & & \sin(nx) \\ & + & \\ 1 & & -\frac{1}{n}\cos(nx) \\ & - & \\ 0 & & -\frac{1}{n^2}\sin(nx) \end{array}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cdot\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\sin(nx) - x\cdot\frac{1}{n}\cos(nx)\right]_{x=-\pi}^{x=0}$$

$$= \frac{1}{\pi}\left[0 - \left(0 - (-\pi)\left(\frac{1}{n}\right)\cos(-n\pi)\right)\right] = \frac{1}{\pi}(-\pi)\left(\frac{1}{n}\right)\cos(n\pi)$$

$$= \frac{-1}{n}(-1)^n.$$

$$\therefore \boxed{f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\cdot\frac{1}{n^2}\left(1-(-1)^n\right)\cos(nx) - \frac{1}{n}\cdot(-1)^n\sin(nx)\right)}$$

(7) $f(x) = \begin{cases} 0 & -\pi \le x < 0 \\ x & 0 \le x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \pi/2.$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cdot\cos(nx)\,dx$$

u		dv
x	$+$	$\cos(nx)$
1	$-$	$1/n\ \sin(nx)$
0		$-1/n^2\ \cos(nx)$

$$= \frac{1}{\pi}\Big[\ 1/n\ x\cdot\sin(nx) + 1/n^2\cos(nx)\Big]_{x=0}^{x=\pi}$$

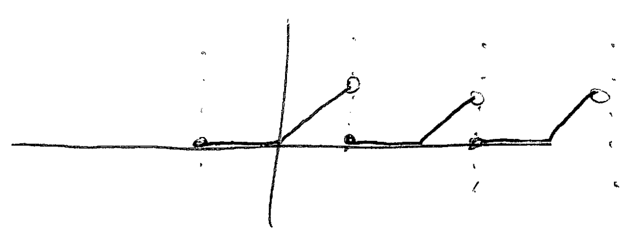
$$= \frac{1}{\pi}\Big[\frac{1}{n^2}\cos(n\pi) - 1/n^2\Big] = \frac{1}{\pi}\cdot 1/n^2\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\cdot\sin(nx)\,dx = \frac{1}{\pi}\Big[(x)\left(-\frac{1}{n}\right)\cos(nx) + \frac{1}{n^2}\sin(nx)\Big]_{x=0}^{x=\pi}$$

$$= \frac{1}{\pi}\Big[-\frac{\pi}{n}\cos(n\pi)\Big] = \left(-\frac{1}{n}\right)(-1)^n = \frac{(-1)^{n+1}}{n}$$

$$\therefore\ \boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\Big[\frac{1}{\pi}\cdot\frac{1}{n^2}\left((-1)^n - 1\right)\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\Big]}$$

f IS CONTINUOUS AND f' IS PIECEWISE CONTINUOUS $\Rightarrow$ FOURIER SERIES CONVERGES TO $f(x)$ ON $(-\pi, \pi)$



(9) $$f(x) = \begin{cases} 0 & -1 \le x < 0 \\ x^2 & 0 \le x < 1 \end{cases}$$

$$a_0 = \frac{1}{1} \cdot \int_{-1}^{1} f(x)\,dx = \int_{0}^{1} x^2\,dx = \frac{1}{3}$$

$$a_n = \int_{0}^{1} x^2 \cos(n\pi x)\,dx$$

$$\begin{array}{ccc} \underline{u} & & \underline{dv} \\ x^2 & & \cos(n\pi x) \\ & + & \\ 2x & & \frac{1}{n\pi}\sin(n\pi x) \\ & - & \\ 2 & & -\left(\frac{1}{n\pi}\right)^2 \cos(n\pi x) \\ & + & \\ 0 & & -\left(\frac{1}{n\pi}\right)^3 \sin(n\pi x) \end{array}$$

$$= \left[\frac{1}{n\pi} x^2 \sin(n\pi x) + \frac{1}{(n\pi)^2} 2x \cos(n\pi x) - \frac{2}{(n\pi)^3} \sin(n\pi x) \right]_{x=0}^{x=1}$$

$$= \frac{2}{(n\pi)^2} \cos(n\pi) = \frac{2}{(n\pi)^2} (-1)^n.$$

$$b_n = \int_{0}^{1} x^2 \sin(n\pi x)\,dx$$

$$\begin{array}{ccc} \underline{u} & & \underline{dv} \\ x^2 & & \sin(n\pi x) \\ & + & \\ 2x & & -\frac{1}{n\pi}\cos(n\pi x) \\ & - & \\ 2 & & -\frac{1}{(n\pi)^2} \sin(n\pi x) \\ & + & \\ 0 & & +\frac{1}{(n\pi)^3} \cos(n\pi x) \end{array}$$

$$= \left[-\frac{x^2}{n\pi} \cos(n\pi x) + \frac{2x}{(n\pi)^2} \sin(n\pi x) + \frac{2}{(n\pi)^3} \cos(n\pi x) \right]_{x=0}^{x=1}$$

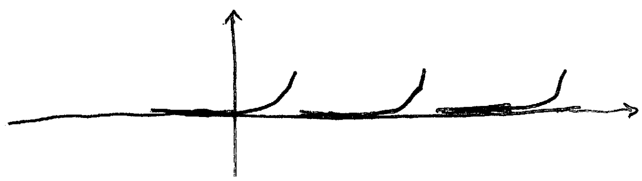
$$= -\frac{1}{n\pi}\cos(n\pi) + \frac{2}{(n\pi)^3}\cos(n\pi) - \frac{2}{(n\pi)^3} = \frac{2}{(n\pi)^3}\left((-1)^n - 1\right) + \frac{(-1)^{n+1}}{n\pi}$$

$$\therefore \boxed{f(x) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}(-1)^n \cos(n\pi x) + \left[\frac{2}{(n\pi)^3}\left[(-1)^n - 1\right] + \frac{(-1)^{n+1}}{n\pi} \right] \sin(n\pi x)}$$

(9) (CONTINUED)

f AND f' ARE PIECEWISE CONTINUOUS

$\Rightarrow$ f HAS CONVERGENT FOURIER SERIES :



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(10) $y'' + \omega^2 y = \sin(nt)$, $y(0) = 0$, $y'(0) = 0$.

SUPPOSE $n^2 \neq \omega^2$ $\Rightarrow$ $y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + y_p$

$y_p = A\cos(nt) + B\sin(nt)$ $\Rightarrow$ $y_p' = -nA\sin(nt) + nB\cos(nt)$

$\Rightarrow y_p'' = -n^2 A \cos(nt) - n^2 B \sin(nt)$

$\Rightarrow y_p'' + \omega^2 y_p = A\cos(nt)\cdot(\omega^2 - n^2) + B\sin(nt)(\omega^2 - n^2) = \sin(nt)$

$\Rightarrow A(\omega^2 - n^2) = 0$ AND $B(\omega^2 - n^2) = 1$

$\Rightarrow A = 0$ $\qquad B = \frac{1}{\omega^2 - n^2}$

$\Rightarrow y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) + \frac{1}{\omega^2 - n^2}\sin(nt)$

$y(0) = c_1 = 0$

$y'(t) = \omega c_2 \cos(\omega t) + \left(\frac{1}{\omega^2 - n^2}\right) n \cos(nt)$

$\Rightarrow y'(0) = 0 = \omega c_2 + \frac{n}{\omega^2 - n^2}$ $\Rightarrow c_2 = \frac{-n}{\omega}\cdot\left(\frac{1}{\omega^2 - n^2}\right)$

$\therefore$ $\boxed{y(t) = \frac{1}{\omega^2 - n^2}\left[\sin(nt) - \frac{n}{\omega}\sin(\omega t)\right] \quad \text{FOR } \omega^2 \neq n^2}$

(10) IF $\omega^2 = n^2$ WE GET <u>RESONANCE</u>.

WE STILL HAVE $y(t) = c_1 \cos(nt) + c_2 \sin(nt) + y_p$

BUT NOW WE GUESS $y_p = t\left(A\cos(nt) + B\sin(nt)\right)$

$$y_p' = \left(A\cos(nt) + B\sin(nt)\right) + t\left[-nA\sin(nt) + nB\cos(nt)\right]$$

$\Rightarrow$

$$y_p'' = \left[-nA\sin(nt) + nB\cos(nt)\right] + \left[-nA\sin(nt) + nB\cos(nt)\right]$$
$$+ t\left[-n^2 A \cancel{\sin}\overset{\cos}{}(nt) - n^2 B\sin(nt)\right]$$

$$y_p'' + n^2 y_p = t\,\cancel{}\left(-n^2 A\cos(nt) + n^2 A\cos(nt) - n^2 B\sin(nt) + n^2 B\sin(nt)\right)$$

$$-2nA\sin(nt) + 2nB\cos(nt) = \sin(nt)$$

$\Rightarrow 2nB = 0$ AND $-2nA = 1$

$\Rightarrow B = 0$ $\quad A = \frac{-1}{2n}$

$$y(t) = c_1\cos(nt) + c_2\sin(nt) - \frac{1}{2n}t\cos(nt)$$

$$y(0) = 0 = c_1$$

$$y'(t) = nc_2\cos(nt) - \frac{1}{2n}\cos(nt) + \frac{1}{2n}t\sin(nt)$$

$$y'(0) = 0 = nc_2 - \frac{1}{2n} \Rightarrow c_2 = \frac{1}{2n^2}.$$

$\therefore$ $$\boxed{y(t) = \frac{1}{2n^2}\sin(nt) - \frac{1}{2n}t\cos(nt)}$$